Reply to Comment on "How skew distributions emerge in evolving systems"

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Abstract. - There is no abstract.

In the preceding comment [1], the author claims that the validity of our letter [2] is limited to the unconfined systems, not applicable to the diffusive systems of material growth with grain boundaries. The main criticism is that we made a very special assumption in the equation of evolution for the number N of sites, $dN/dt = rN^{\chi}$ with $\chi = 1$, which is not applicable to the diffusive growth with χ in general differing from unity and depending on the dimension d of the system. In a series of papers cited in comment [1], Gadomski and his collaborators built a model for grain growth, which was derived directly from the general Fokker-Planck equation. Basically, they started from the following set of Fickian diffusion equations (shown, for example, in reference 2 of comment [1]):

$$\frac{\partial}{\partial t} f(v,t) = -\frac{\partial}{\partial v} J(v,t)$$
$$J(v,t) = -D(v)\frac{\partial}{\partial v} f(v,t), \qquad (1)$$

where v is the volume of the grain, $f(v,t)$ the *distribution* of the grain volume at time t, and $J(v,t)$ is the current. Particles inside the grain can move out of the boundaries and those outside, which are in a different phase, can migrate inside. They also postulated that all the details of grain growth is incorporated phenomenologically in the diffusion coefficient $D(v)$, which usually takes a constant value in the well-known example of heat diffusion.

Since $D(v)$ has the unit of a surface (area) divided by time, the postulated power-law form $D(v) \propto v^{\alpha}$, with v having the dimension of the volume, imposes obviously that $\alpha = (d-1)/d$. Physically, this simply means that the diffusion process is proportional to the surface area of the grain. The grain can only grow in time, leading the peak of the distribution to shift toward large values of the volume v. On the other hand, our system in [2] is represented by a set of N independent sites, the ith of which is assigned a certain height (or size) x_i. Each quantity grows independently according to its own dynamics: While the available number of sites N grows exponentially as $dN/dt = rN$, each site is represented by the height x_i whose dynamics depends on the growth rate λ and the amount b.

Note that the probability distribution $f(x,t)$, representing the probability for the height of a site to take given value x in the system at time t, is always normalized. In contrast, from equation (24) in reference 2 of comment [1], it is understood that the integral of the Fickian distribution is not unity but dependent on time. Here the authors assumed that this integral corresponds to the number of microdomains, which yields the growth rate equation $dN/dt \propto N^{\chi}$ with $\chi = 2 + d$. This is indeed different from our model, where we have adopted the correct normalization. Moreover, in our model, there is an intrinsic dynamics for the number of available sites, which is different from the one for the height at each site. However, in references 2, 4, and 5 of comment [1], the diffusive equation (1), or its variations, contains the dynamics of only one grain, or of a *mean-field* average of a set of grains, dealing with only one size variable v. Accordingly, there is no way in the single equation (1) to compute an analog of N or to distinguish the individual size growth of the grains and the number of grains produced. In particular, it is not possible to retrieve the number $N(t)$ of sites, reversely from the distribution $f(x,t)$ in equation (10) of our letter [2], unless all microscopic details are available. It is therefore not clear that the average of the distribution $f(v,t)$ represents the number of grains at given time t, because of the normalization

$$\int dv\, f(v,t) = 1, \tag{2}$$

which must hold at any time for a distribution function.

If the distribution $f(v,t)$ were normalized, then the average volume given by the first moment, computed in equation (25) in reference 2 of comment [1], would presumably grow as $t^{d/(d+1)}$ instead of $t^{(d-1)/(d+1)}$. The reason is that all the details concerning the dynamics of grain production and individual grain growth are hidden in the phenomenological equation (1), namely, in the diffusive term $D(v)$ or in drift term (see, for example, equation (3) in reference 4 of comment [1]). Only the statistics regarding the average of the grain volume is available. We would need otherwise another set of equations for the precise control of the interaction between grains, thus a variable other than v, for example, N, so that we would define the distribution function $f(v,N,t)$.

Our diffusive "Fickian" equation (10) of our letter [2] includes both dynamics characterized simply by parameters r for the number of sites and b and λ for the height growth process, all in the functional form derived from a *microscopic* master equation. The criticality is perfectly controlled by these parameters. Changing the dynamics or the individual transition rates of our model can easily be implemented without making any assumption on the power-law form of the coefficients appearing in the Fokker-Planck equation.

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REFERENCES

[1] Gadomski A., preceding comment.
[2] Choi M.Y., Choi H., Fortin J.-Y., and Choi J., *Europhys. Lett.*, **85** (2009) 30006.